An Equilibrium Proof to Minimum Surface Equation

(The passage is auto-translated from Chinese.)

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1 Problem Formulation And Notations

In physics, the minimum energy or action means equilibrium and real motion, which is how people transition from classical mechanics to analytical mechanics. In analytical mechanics, energy extremes replace force balance, and Lagrange equations replace Newton's second law. Therefore, every extreme value problem contains an equivalent "classical mechanics viewpoint". From this perspective, the goal of this article is to derive the minimal surface equation from the force balance.

First, I will review the variational derivation of minimal surfaces covered in class, where starting from the variation of the area functional being zero, one can derive that the mean curvature of a minimal surface is zero. Instead of using the normal variation approach found in textbooks, I followed the derivation method presented in class. Next, I derived the minimal surface equation starting from the condition that the mean curvature is zero. Finally, from the perspective of force equilibrium, I re-derived the conditions that minimal surfaces must satisfy—partly inspired by a note I wrote two years ago.

In this article I adopt the following symbols and terminology:

- The bold letters $\boldsymbol{r}$ represent vectors, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ represent the inner product of vector $\boldsymbol{x}, \boldsymbol{y}$, $\boldsymbol{x} \times \boldsymbol{y}$ represent the cross product of vector $\boldsymbol{x}, \boldsymbol{y}$, and $(x, y, z) = \langle \boldsymbol{x} \times \boldsymbol{y}, \boldsymbol{z} \rangle$ represent the mixed product of vector $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

- If the surface Σ in $\mathbb{R}^3$ has a parametric representation: $\boldsymbol{r} = \boldsymbol{r}(u, v) = (u, v, z(u, v))$, then Σ is said to be the graph of the function $z(u, v)$.
- $C_c^\infty(D)$ represents the space of compactly supported C^∞ differentiable continuous functions on the region D .

Unless otherwise specified, only differentiable regular surfaces are considered in this article.

2 Variational Method For Minimum Surface

Consider a surface Σ in $\mathbb{R}^3$ with parametric representation: $\boldsymbol{r} = \boldsymbol{r}(u, v) : \overline{D} \to \mathbb{R}^3$, where the area $D \subset \mathbb{R}^2$ is bounded. The boundary of the surface is given by $\boldsymbol{r}|_{\partial D}$. A minimal surface is a surface that makes the surface extremely minimal given this boundary.

The surface area element is $dS = \langle \boldsymbol{r}_u \times \boldsymbol{r}_v, \boldsymbol{n} \rangle dudv = |\boldsymbol{r}_u \times \boldsymbol{r}_v| dudv$, so the area functional is

$$S(\boldsymbol{r}) = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v| dudv \tag{1}$$

$\boldsymbol{f} : \overline{D} \to \mathbb{R}^3$ is a function that is differentiable by C^∞ , and $\boldsymbol{f}|_{\partial D} \equiv 0$, let all such $\boldsymbol{f}$ form the set F . Next, consider the function with ε as the parameter

$$S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f}) = \iint_D |(\boldsymbol{r}_u + \varepsilon \boldsymbol{f}_u) \times (\boldsymbol{r}_v + \varepsilon \boldsymbol{f}_v)| dudv \tag{2}$$

Then for any $\boldsymbol{f} \in F$, $S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f})$ takes the extreme value at $\varepsilon = 0$. So

$$\left. \frac{d}{d\varepsilon} S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f}) \right|_{\varepsilon=0} = 0, \ \forall \boldsymbol{f} \in F \tag{3}$$

In addition, we can calculate

$$\begin{aligned} \left. \frac{d}{d\varepsilon} S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f}) \right|_{\varepsilon=0} &= \iint_D \left. \frac{d}{d\varepsilon} |(\boldsymbol{r}_u + \varepsilon \boldsymbol{f}_u) \times (\boldsymbol{r}_v + \varepsilon \boldsymbol{f}_v)| \right|_{\varepsilon=0} dudv \\ &= \iint_D \left[\langle \boldsymbol{f}_u \times \boldsymbol{r}_v, \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} \rangle + \langle \boldsymbol{r}_u \times \boldsymbol{f}_v, \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} \rangle \right] dudv \\ &= \iint_D \left[(\boldsymbol{f}_u, \boldsymbol{r}_v, \boldsymbol{n}) + (\boldsymbol{r}_u, \boldsymbol{f}_v, \boldsymbol{n}) \right] dudv \end{aligned} \tag{4}$$

Considering

$$(\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n})_u = (\boldsymbol{f}_u, \boldsymbol{r}_v, \boldsymbol{n}) + (\boldsymbol{f}, \boldsymbol{r}_{uv}, \boldsymbol{n}) + (\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n}_u) \tag{5}$$

$$(\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n})_v = (\boldsymbol{r}_{uv}, \boldsymbol{f}, \boldsymbol{n}) + (\boldsymbol{r}_u, \boldsymbol{f}_v, \boldsymbol{n}) + (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n}_v) \tag{6}$$

Add (5) and (6) and transpose to get

$$(\boldsymbol{f}_u, \boldsymbol{r}_v, \boldsymbol{n}) + (\boldsymbol{r}_u, \boldsymbol{f}_v, \boldsymbol{n}) = \big[(\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n})_u + (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n})_v\big] - \big[(\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n}_u) + (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n}_v)\big] \tag{7}$$

Substitute (7) into (4) to get

$$\begin{aligned}
&\frac{d}{d\varepsilon} S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f})\Big|_{\varepsilon=0} \\
&= \iint_D \big[(\boldsymbol{f}_u, \boldsymbol{r}_v, \boldsymbol{n}) + (\boldsymbol{r}_u, \boldsymbol{f}_v, \boldsymbol{n})\big] dudv \\
&= \iint_D \big[(\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n})_u + (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n})_v\big] dudv - \iint_D \big[(\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n}_u) + (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n}_v)\big] dudv \\
&= \int_{\partial D} (\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n}) dv - (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n}) du - \iint_D \big[(\boldsymbol{r}_v, \boldsymbol{n}_u, \boldsymbol{f}) - (\boldsymbol{r}_u, \boldsymbol{n}_v, \boldsymbol{f})\big] dudv
\end{aligned} \tag{8}$$

The last equal sign uses the Stokes formula. Due to $\boldsymbol{f}|_{\partial D} \equiv 0$, we have

$$\int_{\partial D} (\boldsymbol{f}, \boldsymbol{r}_v, \boldsymbol{n}) dv - (\boldsymbol{r}_u, \boldsymbol{f}, \boldsymbol{n}) du = 0$$

So (8) becomes

$$\frac{d}{d\varepsilon} S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f})\Big|_{\varepsilon=0} = \iint_D \big[(\boldsymbol{r}_u, \boldsymbol{n}_v, \boldsymbol{f}) - (\boldsymbol{r}_v, \boldsymbol{n}_u, \boldsymbol{f})\big] dudv \tag{9}$$

Considering the Weingarten transformation under the (u, v) parameter

$$-\begin{bmatrix} \boldsymbol{n}_u \\ \boldsymbol{n}_v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{bmatrix} \tag{10}$$

(9) becomes

$$\begin{aligned}
\frac{d}{d\varepsilon} S_\varepsilon(\boldsymbol{r} + \varepsilon \boldsymbol{f})\Big|_{\varepsilon=0} &= \iint_D \big[(\boldsymbol{r}_u, c\boldsymbol{r}_u + d\boldsymbol{r}_v, \boldsymbol{f}) - (\boldsymbol{r}_v, a\boldsymbol{r}_u + b\boldsymbol{r}_v, \boldsymbol{f})\big] dudv \\
&= \iint_D (d + a)(\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{f}) dudv \\
&= \iint_D 2H \langle \boldsymbol{r}_u \times \boldsymbol{r}_v, \boldsymbol{f} \rangle dudv \\
&= 0, \ \forall \boldsymbol{f} \in F
\end{aligned} \tag{11}$$

Considering that our surface is C^∞ , $\boldsymbol{r}$ is also infinitely differentiable, and we only consider regular surfaces, so $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is infinitely differentiable and not equal to zero. Therefore, for any infinitely differentiable $f : \overline{D} \to \mathbb{R}, f|_{\partial D} = 0$, the set of such functions f is denoted as F_1 , and obviously $C_c^\infty(D) \subset F_1$. Let $\boldsymbol{f}' = (\boldsymbol{r}_u \times \boldsymbol{r}_v / |\boldsymbol{r}_u \times \boldsymbol{r}_v|^2) \cdot f$, then we have $\boldsymbol{f}' \in F$. Substituting into (11) , we get

$$\iint_D 2H \cdot f dudv = 0, \ \forall f \in C_c^\infty(D) \subset F_1 \tag{12}$$

Finally, according to the standard idea of variational method, since $C_c^\infty(D)$ is dense in $L^1(D)$, $\forall h \in L^2(D) \subset L^1(D)$, there exists a list of functions $\{h_n\}$ in $C_c^\infty(D)$ that converge to h according to the norm. Since $\overline{D} \subset \mathbb{R}^2$ is a bounded closed set, it is a compact set, so $H \in C^\infty(\overline{D})$ is bounded, and D has a finite measure, so we know $H \in L^2(D) \subset L^1(D)$. From this, we can get $G(h) = \int_D 2Hhdudv$ to be a bounded linear functional (and thus continuous).

Since $\{h_n\}$ converges to h norm-wise, $\{h_n\}$ converges weakly to h , resulting in $G(h_n) \to G(h)$, which is $G(h) = 0$. In $L^2(D)$, we have $\forall h, \langle h, H \rangle = 0$, which leads to $H = 0, \ a.s.$, and H is continuous, so we have $H = 0$.

3 An Equilibrium Method For Minimum Surface

In the derivation of this part, we only consider the surface as the graph of a function. At this time, the surface Σ has a parameter representation: $\boldsymbol{r}(u, v) = (u, v, z(u, v))$.

3.1 $H = 0$ Derive minimal surface equations

Next, we will derive the equation of the minimal surface from the condition of the mean curvature $H = 0$ of the minimal surface. First, according to the parameter representation of $\boldsymbol{r}$, we can find

$$\boldsymbol{r}_u = (1, 0, z_u); \ \boldsymbol{r}_v = (0, 1, z_v); \tag{13}$$

$$\boldsymbol{r}_{uu} = (0, 0, z_{uu}); \ \boldsymbol{r}_{uv} = (0, 0, z_{uv}); \ \boldsymbol{r}_{vv} = (0, 0, z_{vv}); \tag{14}$$

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{(-z_u, -z_v, 1)}{\sqrt{1 + z_u^2 + z_v^2}}. \tag{15}$$

From this we can get

$$E = 1 + z_u^2;\ F = z_u z_v;\ G = 1 + z_v^2; \tag{16}$$

$$L = \frac{z_{uu}}{\sqrt{1 + z_u^2 + z_v^2}};\ M = \frac{z_{uv}}{\sqrt{1 + z_u^2 + z_v^2}};\ L = \frac{z_{vv}}{\sqrt{1 + z_u^2 + z_v^2}}; \tag{17}$$

So we can find the mean curvature

$$H = \frac{1}{2(1 + z_u^2 + z_v^2)^{\frac{3}{2}}} \cdot [z_{uu}(1 + z_v^2) - 2z_{uv}z_u z_v + z_{vv}(1 + z_u^2)]. \tag{18}$$

The minimal surface equation can be derived from the mean curvature $H = 0$ of the minimal surface

$$z_{uu}(1 + z_v^2) - 2z_{uv}z_u z_v + z_{vv}(1 + z_u^2) = 0 \tag{19}$$

This is the necessary and sufficient condition for $H = 0$. Next, I will derive the minimal surface equation from the force analysis method.

3.2 An Equilibrium Proof

In physics, the minimum energy always corresponds to the equilibrium state. Therefore, consider a weightless soap film. Its equilibrium state under a fixed boundary is the state where its surface energy is minimal. Since the surface energy is proportional to the surface area of the soap film, the minimum surface energy is equivalent to the minimal surface. Let the surface energy $E_\sigma = S$, where S is the surface area, then the surface tension is proportional to the length of the edge of the patch, and the direction is perpendicular to the edge of the patch and outward.

Consider a soap film on region D , whose boundary $\boldsymbol{r}|_{\partial D}$ is fixed. Establish a right-hand system $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ with plane (u, v) , and consider a surface $\Delta' \subset \Sigma$ of a soap film. Let l' be the boundary of Δ' , then let Δ be the projection of Δ' on plane (u, v) , and let l be the boundary of Δ . Then the resultant force of surface tension can be written as The projection of

$$\boldsymbol{F}_\sigma = \oint_{l'} d\boldsymbol{l}' \times \boldsymbol{n}. \tag{20}$$

in the direction of $\boldsymbol{k}$ is

$$\boldsymbol{F}_{\sigma z} = \oint_{l'} \langle d\boldsymbol{l}' \times \boldsymbol{n}, \boldsymbol{k} \rangle = \oint_{l'} (d\boldsymbol{l}', \boldsymbol{n}, \boldsymbol{k}) \tag{21}$$

According to the force balance in the direction of $\boldsymbol{k}$, we have

$$\boldsymbol{F}_{\sigma z} = \oint_{l'} (d\boldsymbol{l}', \boldsymbol{n}, \boldsymbol{k}) = 0. \tag{22}$$

Since l is the projection of l' , we have the following relationship

$$d\boldsymbol{l}' = d\boldsymbol{l} + \langle d\boldsymbol{l}, (z_u, z_v, 0) \rangle \cdot \boldsymbol{k}. \tag{23}$$

Substituting (22) into (23) , we get

$$\begin{aligned} \oint_{l'} (d\boldsymbol{l}', \boldsymbol{n}, \boldsymbol{k}) &= \oint_{l} ([d\boldsymbol{l} + \langle d\boldsymbol{l}, (z_u, z_v, 0) \rangle \cdot \boldsymbol{k}], \boldsymbol{n}, \boldsymbol{k}) \\ &= \oint_{l} (d\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{k}) = 0 \end{aligned} \tag{24}$$

Let $\theta = \arctan(\sqrt{1 + z_u^2 + z_v^2})$ be the angle between a certain face element and the (u, v) plane, then we have

$$\boldsymbol{n} = \boldsymbol{k} \cos\theta - \frac{(z_u, z_v, 0)}{\sqrt{z_u^2 + z_v^2}} \sin\theta \tag{25}$$

Stretch the (u, v) plane upward to any height h to form a cylinder, let the side of the cylinder be Σ_0 , then (24) can be transformed into

$$\begin{aligned} 0 = \oint_{l} (d\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{k}) &= \frac{1}{h} \oint_{l} \langle h \cdot (\boldsymbol{k} \times d\boldsymbol{l}), \boldsymbol{n} \rangle \\ &= -\frac{1}{h} \iint_{\Sigma_0} \langle \boldsymbol{n}, d\boldsymbol{S} \rangle \end{aligned} \tag{26}$$

where $d\boldsymbol{S}$ represents the face element on the side of the cylinder.

Substituting (25) into (26) yields

$$\iint_{\Sigma_0} \langle \frac{(z_u, z_v, 0)}{\sqrt{z_u^2 + z_v^2}} \sin\theta, d\boldsymbol{S} \rangle = 0 \tag{27}$$

Considering that $(z_u, z_v, 0)$ is parallel to Σ_0 , we can replace the cylinder side Σ_0 with the entire cylinder surface Σ_1 in the integral, so we get

$$
\begin{aligned}
0 &= \iint_{\Sigma_1} \langle \frac{(z_u, z_v, 0)}{\sqrt{z_u^2 + z_v^2}} \sin\theta, d\boldsymbol{S} \rangle \\
&= \iiint_V \nabla \cdot \left(\frac{(z_u, z_v, 0)}{\sqrt{z_u^2 + z_v^2}} \sin\theta \right) dV \\
&= \iiint_V \nabla \cdot \left(\frac{(z_u, z_v, 0)}{\sqrt{1 + z_u^2 + z_v^2}} \right) dV
\end{aligned}
\tag{28}
$$

This is true for any cylinder V , where dV is the volume element and $\nabla = (\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial h})$ is the Nabla operator. From this, we can conclude that

$$
\nabla \cdot \left(\frac{(z_u, z_v, 0)}{\sqrt{1 + z_u^2 + z_v^2}} \right) = 0 \tag{29}
$$

can be expanded to yield

$$
\begin{aligned}
0 =& \nabla \cdot \left(\frac{(z_u, z_v, 0)}{\sqrt{1 + z_u^2 + z_v^2}} \right) \\
=& \frac{1}{(1 + z_u^2 + z_v^2)^{\frac{3}{2}}} \cdot [z_{uu}(1 + z_v^2) - 2z_{uv}z_u z_v + z_{vv}(1 + z_u^2)] \\
=& 2H
\end{aligned}
\tag{30}
$$

, which means that we can also deduce the conditions and equations for the minimal surface $H = 0$.

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微分几何结课小报告

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1 问题引入与符号约定

在物理中，能量或作用量极小意味着平衡状态和真实运动，由此人们从经典力学过渡到了分析力学。在分析力学中，能量极值取代了受力平衡，拉格朗日方程取代了牛顿第二定律，因此每一种极值问题都包含了一种等价的 “经典力学观点”。从这个角度出发，本文的目标是从受力平衡出发，推出极小曲面方程。

首先，我将回顾课堂中讲过的极小曲面的变分法推导，即从面积泛函的变分为 0 出发可以导出极小曲面的平均曲率为 0。我没有采取书本上进行的法向变分，而是按照上课时的思路进行了推导。接下来，我从平均曲率为 0 出发，推导了极小曲面方程。最后，我从受力平衡的角度，重新推导了极小曲面满足的条件——其中部分灵感来自我两年前写过的一篇笔记。

在本文中我采取如下的符号和术语：

- 粗体字母 $\mathbf{r}$ 表示向量，$\langle \mathbf{x}, \mathbf{y} \rangle$ 表示向量 $\mathbf{x}, \mathbf{y}$ 的内积，$\mathbf{x} \times \mathbf{y}$ 表示向量 $\mathbf{x}, \mathbf{y}$ 的叉乘，$(x, y, z) = \langle \mathbf{x} \times \mathbf{y}, \mathbf{z} \rangle$ 表示向量 $\mathbf{x}, \mathbf{y}, \mathbf{z}$ 的混合积。
- 若 $\mathbb{R}^3$ 中的曲面 Σ 具有参数表示：$\mathbf{r} = \mathbf{r}(u, v) = (u, v, z(u, v))$，则称 Σ 是函数 $z(u, v)$ 的图。
- $C_c^\infty(D)$ 表示区域 D 上紧支集的 C^∞ 可微连续函数空间。

如未特殊说明，在文章中只考虑 C^∞ 可微的正则曲面。

2 极小曲面的变分法推导

考虑 $\mathbb{R}^3$ 中的曲面 Σ 具有参数表示：$\mathbf{r}=\mathbf{r}(u,v):\overline{D}\to\mathbb{R}^3$，其中区域 $D\subset\mathbb{R}^2$ 且有界。曲面的边界 $\mathbf{r}|_{\partial D}$ 给定。极小曲面为在这个给定边界下，使曲面面积极小的曲面。

曲面的面积元 $dS=\langle\mathbf{r}_u\times\mathbf{r}_v,\mathbf{n}\rangle dudv=|\mathbf{r}_u\times\mathbf{r}_v|dudv$，因此，面积泛函为

$$S(\mathbf{r})=\iint_D|\mathbf{r}_u\times\mathbf{r}_v|dudv \tag{1}$$

$\mathbf{f}:\overline{D}\to\mathbb{R}^3$ 为 C^∞ 可微的函数，且 $\mathbf{f}|_{\partial D}\equiv 0$，令所有这样的 $\mathbf{f}$ 组成集合 F。接下来考虑以 ε 为参数的函数

$$S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})=\iint_D|(\mathbf{r}_u+\varepsilon\mathbf{f}_u)\times(\mathbf{r}_v+\varepsilon\mathbf{f}_v)|dudv \tag{2}$$

则有对任意 $\mathbf{f}\in F$，$S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})$ 在 $\varepsilon=0$ 处取极值。因此有

$$\frac{d}{d\varepsilon}S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})\bigg|_{\varepsilon=0}=0,\ \forall\mathbf{f}\in F \tag{3}$$

此外，我们可以计算得

$$\begin{aligned}\frac{d}{d\varepsilon}S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})\bigg|_{\varepsilon=0}&=\iint_D\frac{d}{d\varepsilon}|(\mathbf{r}_u+\varepsilon\mathbf{f}_u)\times(\mathbf{r}_v+\varepsilon\mathbf{f}_v)|\bigg|_{\varepsilon=0}dudv\\&=\iint_D\left[\langle\mathbf{f}_u\times\mathbf{r}_v,\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\rangle+\langle\mathbf{r}_u\times\mathbf{f}_v,\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\rangle\right]dudv\\&=\iint_D\big[(\mathbf{f}_u,\mathbf{r}_v,\mathbf{n})+(\mathbf{r}_u,\mathbf{f}_v,\mathbf{n})\big]dudv\end{aligned} \tag{4}$$

再考虑到

$$(\mathbf{f},\mathbf{r}_v,\mathbf{n})_u=(\mathbf{f}_u,\mathbf{r}_v,\mathbf{n})+(\mathbf{f},\mathbf{r}_{uv},\mathbf{n})+(\mathbf{f},\mathbf{r}_v,\mathbf{n}_u) \tag{5}$$

$$(\mathbf{r}_u,\mathbf{f},\mathbf{n})_v=(\mathbf{r}_{uv},\mathbf{f},\mathbf{n})+(\mathbf{r}_u,\mathbf{f}_v,\mathbf{n})+(\mathbf{r}_u,\mathbf{f},\mathbf{n}_v) \tag{6}$$

将(5)与(6)相加并移项得

$$(\mathbf{f}_u,\mathbf{r}_v,\mathbf{n})+(\mathbf{r}_u,\mathbf{f}_v,\mathbf{n})=\big[(\mathbf{f},\mathbf{r}_v,\mathbf{n})_u+(\mathbf{r}_u,\mathbf{f},\mathbf{n})_v\big]-\big[(\mathbf{f},\mathbf{r}_v,\mathbf{n}_u)+(\mathbf{r}_u,\mathbf{f},\mathbf{n}_v)\big] \tag{7}$$

将(7)代入(4)得

$$
\begin{aligned}
&\frac{d}{d\varepsilon}S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})\bigg|_{\varepsilon=0}\\
&=\iint_D \big[(\mathbf{f}_u,\mathbf{r}_v,\mathbf{n})+(\mathbf{r}_u,\mathbf{f}_v,\mathbf{n})\big]dudv\\
&=\iint_D \big[(\mathbf{f},\mathbf{r}_v,\mathbf{n})_u+(\mathbf{r}_u,\mathbf{f},\mathbf{n})_v\big]dudv-\iint_D \big[(\mathbf{f},\mathbf{r}_v,\mathbf{n}_u)+(\mathbf{r}_u,\mathbf{f},\mathbf{n}_v)\big]dudv\\
&=\int_{\partial D}(\mathbf{f},\mathbf{r}_v,\mathbf{n})dv-(\mathbf{r}_u,\mathbf{f},\mathbf{n})du-\iint_D \big[(\mathbf{r}_v,\mathbf{n}_u,\mathbf{f})-(\mathbf{r}_u,\mathbf{n}_v,\mathbf{f})\big]dudv
\end{aligned}
\tag{8}
$$

其中最后一个等号用到了斯托克斯公式。由于 $\mathbf{f}|_{\partial D}\equiv 0$，我们有

$$
\int_{\partial D}(\mathbf{f},\mathbf{r}_v,\mathbf{n})dv-(\mathbf{r}_u,\mathbf{f},\mathbf{n})du=0
$$

因此(8)变为

$$
\frac{d}{d\varepsilon}S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})\bigg|_{\varepsilon=0}=\iint_D \big[(\mathbf{r}_u,\mathbf{n}_v,\mathbf{f})-(\mathbf{r}_v,\mathbf{n}_u,\mathbf{f})\big]dudv \tag{9}
$$

考虑到 (u,v) 参数下的 Weingarten 变换

$$
-\begin{bmatrix}\mathbf{n}_u\\ \mathbf{n}_v\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\mathbf{r}_u\\ \mathbf{r}_v\end{bmatrix} \tag{10}
$$

(9)变为

$$
\begin{aligned}
\frac{d}{d\varepsilon}S_\varepsilon(\mathbf{r}+\varepsilon\mathbf{f})\bigg|_{\varepsilon=0}&=\iint_D \big[(\mathbf{r}_u,c\mathbf{r}_u+d\mathbf{r}_v,\mathbf{f})-(\mathbf{r}_v,a\mathbf{r}_u+b\mathbf{r}_v,\mathbf{f})\big]dudv\\
&=\iint_D (d+a)(\mathbf{r}_u,\mathbf{r}_v,\mathbf{f})dudv\\
&=\iint_D 2H\langle\mathbf{r}_u\times\mathbf{r}_v,\mathbf{f}\rangle dudv\\
&=0,\ \forall\mathbf{f}\in F
\end{aligned}
\tag{11}
$$

考虑到我们的曲面是 C^∞ 的，$\mathbf{r}$ 也是无穷次可微的，且我们只考虑正则曲面，因此 $\mathbf{r}_u\times\mathbf{r}_v$ 无穷次可微且不等于零。因此，任意无穷次可微的 $f:\overline{D}\to\mathbb{R}, f|_{\partial D}=0$，这样的函数 f 的集合记作 F_1，显然 $C_c^\infty(D)\subset F_1$。令 $\mathbf{f}'=(\mathbf{r}_u\times\mathbf{r}_v/|\mathbf{r}_u\times\mathbf{r}_v|^2)\cdot f$，则有 $\mathbf{f}'\in F$。代入(11)得

$$
\iint_D 2H\cdot f dudv=0,\ \forall f\in C_c^\infty(D)\subset F_1 \tag{12}
$$

最后则是按照变分法的标准思路，由于 $C_c^\infty(D)$ 在 $L^1(D)$ 中稠密，$\forall h \in L^2(D) \subset L^1(D)$, 存在一列 $C_c^\infty(D)$ 中的函数 $\{h_n\}$ 依范数收敛到 h。由于 $\overline{D} \subset \mathbb{R}^2$ 是有界闭集，因此是紧集，于是 $H \in C^\infty(\overline{D})$ 有界，再加上 D 测度有限，可知 $H \in L^2(D) \subset L^1(D)$。由此可得 $G(h) = \int_D 2Hhdudv$ 为有界线性泛函（从而也是连续的）。

由于 $\{h_n\}$ 依范数收敛到 h，因此 $\{h_n\}$ 弱收敛到 h，从而得到 $G(h_n) \to G(h)$，即 $G(h) = 0$。在 $L^2(D)$ 中，则有 $\forall h, \langle h, H\rangle = 0$，继而推出 $H = 0,\ a.s.$，再加上 H 连续，因此 $H = 0$。

3 极小曲面的受力分析法推导

在这一部分的推导中，我们只考虑曲面是某个函数的图。此时，曲面 Σ 具有参数表示：$\mathbf{r}(u, v) = (u, v, z(u, v))$。

3.1 H = 0 推导极小曲面方程

下面我们将从极小曲面平均曲率 $H = 0$ 的条件出发，推出极小曲面的方程来。首先根据 $\mathbf{r}$ 的参数表示，可以求出

$$\mathbf{r}_u = (1, 0, z_u);\ \mathbf{r}_v = (0, 1, z_v); \tag{13}$$

$$\mathbf{r}_{uu} = (0, 0, z_{uu});\ \mathbf{r}_{uv} = (0, 0, z_{uv});\ \mathbf{r}_{vv} = (0, 0, z_{vv}); \tag{14}$$

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \frac{(-z_u, -z_v, 1)}{\sqrt{1 + z_u^2 + z_v^2}}. \tag{15}$$

由此可以得到

$$E = 1 + z_u^2;\ F = z_u z_v;\ G = 1 + z_v^2; \tag{16}$$

$$L = \frac{z_{uu}}{\sqrt{1 + z_u^2 + z_v^2}};\ M = \frac{z_{uv}}{\sqrt{1 + z_u^2 + z_v^2}};\ L = \frac{z_{vv}}{\sqrt{1 + z_u^2 + z_v^2}}; \tag{17}$$

从而可以求出平均曲率为

$$H = \frac{1}{2(1 + z_u^2 + z_v^2)^{\frac{3}{2}}} \cdot [z_{uu}(1 + z_v^2) - 2z_{uv}z_u z_v + z_{vv}(1 + z_u^2)]. \tag{18}$$

由极小曲面平均曲率 $H = 0$ 可以推出极小曲面方程

$$z_{uu}(1 + z_v^2) - 2z_{uv}z_u z_v + z_{vv}(1 + z_u^2) = 0 \tag{19}$$

这就是 $H=0$ 的充分必要条件。接下来，我将从受力分析法出发，推导出极小曲面方程。

3.2 受力分析法推导极小曲面方程

在物理上，能量极小总是对应了平衡状态，因此考虑一个无重力的肥皂膜，其在固定边界下的平衡状态就是其表面能极小的状态。由于表面能正比于肥皂膜表面积，因此表面能极小等价于极小曲面。令表面能 $E_\sigma = S$，其中 S 为表面积，则表面张力正比于面片边缘长度，方向垂直面片边缘向外。

考虑区域 D 上的肥皂膜，其边界 $\mathbf{r}|_{\partial D}$ 固定。以 (u,v) 平面建立右手系 $(\mathbf{i},\mathbf{j},\mathbf{k})$，考虑一块肥皂膜的面 $\Delta' \subset \Sigma$，我们令 l' 为 Δ' 的边界，则令 Δ 为 Δ' 在 (u,v) 平面上的投影，我们令 l 为 Δ 的边界。那么表面张力的合力可以写成

$$\mathbf{F}_\sigma = \oint_{l'} d\mathbf{l}' \times \mathbf{n}. \tag{20}$$

在 $\mathbf{k}$ 方向的投影为

$$\mathbf{F}_{\sigma z} = \oint_{l'} \langle d\mathbf{l}' \times \mathbf{n}, \mathbf{k} \rangle = \oint_{l'} (d\mathbf{l}', \mathbf{n}, \mathbf{k}) \tag{21}$$

依据 $\mathbf{k}$ 方向受力平衡，我们有

$$\mathbf{F}_{\sigma z} = \oint_{l'} (d\mathbf{l}', \mathbf{n}, \mathbf{k}) = 0. \tag{22}$$

由于 l 是 l' 的投影，我们有如下关系

$$d\mathbf{l}' = d\mathbf{l} + \langle d\mathbf{l}, (z_u, z_v, 0) \rangle \cdot \mathbf{k}. \tag{23}$$

将(22)代入(23)可得

$$\begin{aligned} \oint_{l'} (d\mathbf{l}', \mathbf{n}, \mathbf{k}) &= \oint_{l} ([d\mathbf{l} + \langle d\mathbf{l}, (z_u, z_v, 0) \rangle \cdot \mathbf{k}], \mathbf{n}, \mathbf{k}) \\ &= \oint_{l} (d\mathbf{l}, \mathbf{n}, \mathbf{k}) = 0 \end{aligned} \tag{24}$$

令 $\theta = \arctan(\sqrt{1+z_u^2+z_v^2})$ 为某处面元与 (u,v) 平面的夹角，则有

$$\mathbf{n} = \mathbf{k}\cos\theta - \frac{(z_u, z_v, 0)}{\sqrt{z_u^2+z_v^2}} \sin\theta \tag{25}$$

将 (u,v) 平面向上拉伸任意高度 h 形成柱体, 令柱体的侧面为 Σ_0，则(24)可化为

$$\begin{aligned}0&=\oint_l (d\mathbf{l},\mathbf{n},\mathbf{k})=\frac{1}{h}\oint_l\langle h\cdot(\mathbf{k}\times d\mathbf{l}),\mathbf{n}\rangle\\&=-\frac{1}{h}\iint_{\Sigma_0}\langle\mathbf{n},d\mathbf{S}\rangle\end{aligned}\tag{26}$$

其中 $d\mathbf{S}$ 表示柱体侧面的面元。

将(25)代入(26)可得

$$\iint_{\Sigma_0}\langle\frac{(z_u,z_v,0)}{\sqrt{z_u^2+z_v^2}}\sin\theta,d\mathbf{S}\rangle=0\tag{27}$$

考虑到 $(z_u,z_v,0)$ 与 Σ_0 平行，可以在积分中将柱体侧面 Σ_0 换成柱体整个表面 Σ_1，于是我们得到了

$$\begin{aligned}0&=\iint_{\Sigma_1}\langle\frac{(z_u,z_v,0)}{\sqrt{z_u^2+z_v^2}}\sin\theta,d\mathbf{S}\rangle\\&=\iiint_V\nabla\cdot\left(\frac{(z_u,z_v,0)}{\sqrt{z_u^2+z_v^2}}\sin\theta\right)dV\\&=\iiint_V\nabla\cdot\left(\frac{(z_u,z_v,0)}{\sqrt{1+z_u^2+z_v^2}}\right)dV\end{aligned}\tag{28}$$

对任意柱体 V 均成立，其中 dV 为体积元，$\nabla=(\frac{\partial}{\partial u},\frac{\partial}{\partial v},\frac{\partial}{\partial h})$ 为 Nabla 算子。由此可以得出

$$\nabla\cdot\left(\frac{(z_u,z_v,0)}{\sqrt{1+z_u^2+z_v^2}}\right)=0\tag{29}$$

展开后可得

$$\begin{aligned}0=&\nabla\cdot\left(\frac{(z_u,z_v,0)}{\sqrt{1+z_u^2+z_v^2}}\right)\\=&\frac{1}{(1+z_u^2+z_v^2)^{\frac{3}{2}}}\cdot[z_{uu}(1+z_v^2)-2z_{uv}z_uz_v+z_{vv}(1+z_u^2)]\\=&2H\end{aligned}\tag{30}$$

即同样可以推出极小曲面 $H=0$ 的条件以及极小曲面方程。

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